MATH 203

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REAL ANALYSIS 1

Course outlines

1. preliminaries - properties of Real numbers: Algebraic and topological properties of $\mathbb{R}$. Identity theory, ordering properties Boundedness and other related results.
Sequence of $\mathbb{R}$: Sequence, bounded,
ii. Sequences, subsequence, monotone sequence Cauchy sequence and other related results.
iii Series of $\mathbb{R}$: partial sum of a series Convergent test.

Recall

$\mathbb{N} = \{1, 2, 3 - - - - - \}$

$\mathbb{Z} = \{- - - -2, -1, 0, 1, 2 - - - - \}$

$\mathbb{Q} = \{ a/b : a, b \in \mathbb{Z} \text{ and } b \neq 0 \}$

Irrational $\mathbb{Q}' = \{$

$\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}'$

properties of $\mathbb{R}$

1. Ordering properties or order relation: we define this ordering relation in this following Axioms.

Axiom 1 (Tricotomy)

$\forall\, a, b \in \mathbb{R}$ exactly one of the following must hold

(i) $a = b$ (ii) $a < b$ (iii) $b < a$

Number $a \in \mathbb{R}$ s.t $a > 0$ is positive

Number $b \in \mathbb{R}$ s.t $b < 0$ is Negative

This tricotomy axioms acerts that every real number other than Zero is either negative or positive but NEVER

DATE

Both. Zero is a natural number

* Axiom 2 for all $a,b,c \in \mathbb{R}$ $a<b$ iff $a+c<b+c$

* Axiom 3 $\forall$ $a,b,c \in \mathbb{R}$ if $a>0$ and $b>0$ the $a+b>0$

PROPOSITIONS $\forall$ $a,b \in \mathbb{R}$

i. $a<b$ iff $a-b<0$ iff $b-a>0$

ii. If $a<b$ and $b<c$ then $a<c$

proof

$b-a>0$ \$ $c-b>0$

$b-a+c-b>0$

$c-a>0$

$c>a$

theorem
lemma
proposition
corollary

iii $a^2 \geq 0$ $\forall a \in \mathbb{R}$

proof let $a \in \mathbb{R}$

If $a<0 \Rightarrow -a \geq 0$

$\therefore (-a)\cdot(-a) \geq (a.a) = a^2 > 0$

Also if $a>0$ $a.a>0$ $a^2>0$ //

if $a=0$ $a.a=0$ $a^2=0$ // $a^2 \geq 0$ //

iv ~~if~~ $a>0$ then $1/a > 0$

v. If $a>b>0$ then $1/a < 1/b$

proof by Contradiction

let $a>0$ suppose $1/a \leq 0$

if $1/a = 0 \Rightarrow 1 = a.0 \Rightarrow 1=0$

if $1/a < 0 \Rightarrow 1<0$ by contradiction

Hence if $a>0$ then $1/a > 0$

$a=2$ $b=3$
$a<b$ $a-b<0$
$2<3$ $2-3<0$
$-1<0$
$b-a>0$
$3-2>0$ $1>0$

Absolute Value

$$\forall a \in \mathbb{R} \quad |a| = \begin{cases} a & \text{if } a>0 \\ 0 & \text{if } a=0 \\ -a & \text{if } a<0 \end{cases}$$

Example

$|-5| = -(-5) = 5$

Theorem 1

a,b i. $|a| \geq 0 \ \forall a \in \mathbb{R}$ $|a| = 0$ iff $a = 0$

ii $|a| \leq \varepsilon$ iff $-\varepsilon < a < \varepsilon \ \forall \varepsilon > 0$ $\forall a \in \mathbb{R}$

iii $|ab| = |a||b| \ \forall a,b \in \mathbb{R}$

iv $|a+b| \leq |a| + |b| \ \forall a,b \in \mathbb{R}$

proof

i $|a| \geq 0 \ \forall a \in \mathbb{R}$

$|a| = 0$ iff $a = 0$

if $a = 0$ by definition $|a| = 0$

Case 1 if $a > 0$ $|a| = a > 0$.

$\Rightarrow |a| > 0$ $|a| \geq 0$

case 2 if $a < 0$ $|a| = -a \geq 0$

$\Rightarrow |a| > 0$

$|a| = a$ $|a| = -a$ then

$-|a| \leq a \leq |a|$

proof 2 suppose $|a| \leq \varepsilon$

$\Rightarrow |a| \leq \varepsilon \Rightarrow -\varepsilon \leq -|a| \leq a \leq |a| \leq \varepsilon$

$\Rightarrow -\varepsilon \leq a \leq \varepsilon$

* Conversity $-\varepsilon \leq a \leq \varepsilon$

Case 1 if $a > 0$

$|a| = a \leq \varepsilon \Rightarrow |a| \leq \varepsilon$

Case 2 if $a < 0$ $|a| \leq \varepsilon \ \forall a \in \mathbb{R}$

$|a| = -a \leq \varepsilon \Rightarrow |a| \leq \varepsilon$

proof 3.

$|ab| = |a||b|$

Recall that if $\begin{cases} a > 0 \quad b > 0 \\ ab > 0 \text{ then} \end{cases}$

if $a > 0$ $b > 0$

$|a| = a$ $|b| = b$

$|ab| = ab = |a||b|$

$\therefore |ab| = |a||b|$ //

* case 2 if $a < 0$ $b < 0$ $ab > 0$

show

$|a| = -a$ $|b| = -b$

$|ab| = ab = (-a).(-b) = |a||b|$

$|ab| = |a||b|$

Case 3 $a < 0$ $b > 0$ $ab < 0$

show. $|a| = -a$ $|b| = b$ $|ab| = -ab$

$\Rightarrow (-a)(b) = |a||b|$

$\forall a,b \in \mathbb{R}$ $ab = |a||b|$

Proof 4 Trian

$|a+b| \leq |a| + |b|$, $-|a| \leq a \leq |a|$

$-|b| \leq b \leq |b|$

$-(|a| + |b|) \leq a+b \leq |a| + |b|$

$\Rightarrow |a+b| \leq |a| + |b|$ //

Exercise

find all $x \in \mathbb{R}$ that satisfy the following inequalities

(i) $|4x - 3| \leq 11$

(ii) $|x - 2| > |x + 1|$

DATE

(iii) $|2-5x| \le 7$ (iv) $|x^2-4| < 5$

(v) $|x| + |x+2| < 5$

(vi) $|x| + |x+2| + |2-x| \le 8$

Solution

① $|4x-3| \le 11$

first we find the value of x that makes the absolute value function to be zero

$x = 3/4$ $\qquad 0 \le 11$

then find the when $x > 3/4$

$\Rightarrow |4x-3| = 4x-3 \le 11$

$4x \le 14$

$x \le 7/2$

$\therefore 3/4 \le x \le 7/2$

if $x < 3/4$ then

$|4x-3| = -(4x-3) \le 11$

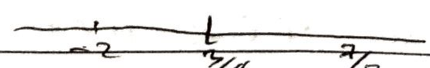
$-4x+3 \le 11$

$-4x \le 8$

$x \ge -2$

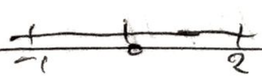
$\therefore$ the range of $x \in \mathbb{R}$ such that $|4x-3| \le 11$ holds are

$-2 \le x \le 3/4$ and $3/4 \le x \le 7/2$



$-2 \le x \le 7/2$ //

② $|x-2| > |x+1|$



if $x > 2$

$|x-2| = x-2$ and $|x+1| = x+1$

$x-2 > x+1$

$-2-1 > 0$

$-3 > 0$ ✗

since there is a contradiction all $x > 2$ will not satisfy the inequality.

if $x < -1$ $|x-2| = -(x-2)$

and $|x+1| = -(x+1)$

$\therefore 2-x > -x-1$

$3 > 0$ True

$\therefore$ All numbers < 1 can satisfy

if $x \in (-1, 2)$ $|x-2| = -(x-2)$

$|x+1| = x+1$

$-(x-2) > x+1$

$2-x > x+1$

$1 > 2x$

$x = 1/2$ All value of x $< 1/2$ will satisfy.

Exercise

① prove that if $b \ne 0$ then $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

② show that $\forall a, b \in \mathbb{R}$

i. $ab \le \frac{1}{2}(a^2+b^2)$

ii. $\frac{(a+b)}{2} \le (a^2+b^2)$

$\{ (1)\ 2\ 2\ (7) \}$
$n=1$

DATE

iii

iii. $\sqrt{ab} \le \frac{1}{2}(a+b)$ for $a, b > 0$

(3) show that $\forall a,b,c,d \in \mathbb{R}$ with $a,b,c,d \ge 0$

(i) $\sqrt{ab}\cdot\sqrt{cd} \le \frac{1}{4}(a^2+b^2+c^2+d^2)$

(ii) $(abcd)^{1/4} \le \frac{1}{4}(a+b+c+d)$

prove that $\forall \varepsilon > 0$ and $a\ x \in \mathbb{R}$ then

$|a-x| < \varepsilon$ iff

$x-\varepsilon < a < x+\varepsilon$

23/1/2018

Mathematical Induction

Inductive property

A set is said to have inductive property if one is in the set $n+1$ is in the set whenever n is in the set.

$1 \in \mathbb{N}$

$n+1 \in \mathbb{N}$ whenever $n \in \mathbb{N}$

$1 \in \mathbb{N}$, $2 = 2+1$

Theorem (mathematical induction)

(i) If $p(n)$ denotes a proposition that depend on n

(ii) $p(1)$ is true

(iii) $p(n+1)$ is true whenever $p(n)$ is true. then $p(k)$ is true for all $k \in \mathbb{N}$

Example

(1) prove that

$1+2+3+\cdots+n = \frac{n}{2}(n+1)$

proof

$p(n) = 1+2+3+\cdots+n = \frac{n}{2}(n+1)$

soln

$p(1) = 1 = \frac{1}{2}(1+1)$

$1 = 1$

$p(1)$ is true

Suppose that $p(n)$ is true i.e $p(n) = \frac{n}{2}(n+1)$ and we show that $p(n+1)$ is true

$p(n+1) = 1+2+3+\cdots+n+n+1$

$p(n+1) = p(n)+n+1$

$\Rightarrow p(n+1) = \frac{n}{2}(n+1)+(n+1)$

$\frac{n(n+1)+2(n+1)}{2}$

$= \frac{(n+1)(n+2)}{2}$ *

$p(n+1) = \frac{n+1}{2}((n+1)+1)$

$\Rightarrow p(n+1)$ is true

Hence $p(k)$ is true $\forall k \in \mathbb{N}$

(2) $1+r+r^2+\cdots+r^n = \frac{1-r^{n+1}}{1-r}$

$p(1) = 1+r = \frac{1-r^{1+1}}{1-r}$

$= \frac{1-r^2}{1-r} = \frac{(1-r)(1+r)}{1-r} = 1+r$

$p(2)$ ...

DATE

Suppose that $p(n)$ is true

i.e $1+r+r^2+\dots+r^n = \frac{1-r^{n+1}}{1-r}$

$\Rightarrow p(n+1) = 1+r+r^2+\dots+r^n+r^{n+1}$

$= \frac{1-r^{n+1}}{1-r} + r^{n+1}$

$\frac{1-r^{n+1}+(1-r)(r^{n+1})}{1-r}$

$\frac{1-\cancel{r^{n+1}}+\cancel{r^{n+1}}-r\,r^{n+1}}{1-r}$

$\frac{1-r^{n+1+1}}{1-r} = \frac{1-r^{n+2}}{1-r}$

Exercise

prove the following by Mathematical Induction

① $1^2+2^2+3^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$

(ii) $1^3+2^3+3^3+\dots+n^3 = \frac{n^2(n+1)^2}{4}$

③ $n! \geq 2^{n-1}$ $\forall n \in \mathbb{N}$

④ $2^n \geq n^2$ $\forall n$

⑤ if $x>0$ then $(1+n)^2 = 1+nx$ $\forall n \in \mathbb{N}$

⑥ $11^n - 8^n$ is a multiple of 3 $\forall n \in \mathbb{N}$

Extended real number

The set $\mathbb{R}$ is extended by including two element denoted by ∞ and $-\infty$

The extended set denoted by $\mathbb{R}^*$ is the set $\mathbb{R}^* = \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$

The following are some definition that partially extended the operation plus and multiplication in $\mathbb{R}$

① $-\infty < \infty$

(ii) $-\infty < a < \infty$ $\forall a \in \mathbb{R}$

(iii) $a+\infty = \infty + a = \infty$

(iv) $a+(-\infty) = -\infty + a = -\infty$ $\forall a \in \mathbb{R}$

(v) $a \cdot \infty = \infty \cdot a = \begin{cases} \infty & \text{if } a>0 \\ -\infty & \text{if } a<0 \end{cases}$

(vi) $a \cdot (-\infty) = \begin{cases} -\infty & \text{if } >0 \\ \infty & \text{if } a<0 \end{cases}$

Intervals

① $(a,b) = \{x \in \mathbb{R}^* : a<x<b\}$ open intervals

(ii) $[a,b] = \{x \in \mathbb{R}^* : a \leq x \leq b\}$ closed interval

(iii) $[a,b) = \{x \in \mathbb{R}^* : a \leq x < b\}$

(iv) $(a,b] = \{x \in \mathbb{R}^* : a < x \leq b\}$

Note

If I is an interval then it has a property that $y_1, y_2 \in I$ and $y_1 < x < y_2$ then $x \in I$

eg $(1, 2)$ $[0, 1)$ $[0, 1]$

DATE

Bounded set

Def: A subset B of $\mathbb{R}$ is said to be bounded from below if $\exists\ \alpha \in \mathbb{R}$ such that

$\alpha \leq x \quad \forall\ x \in B$

$I = (0\ 1) \subset \mathbb{R}$

$\alpha = 0 \leq x \quad \forall\ x \in I$

This show that is bounded below.

- A subset B of $\mathbb{R}$ is said to be bounded from above if $\exists\ \beta \in \mathbb{R}$ such that $\beta \geq x$ $\forall\ x \in B$

- A subset B of $\mathbb{R}$ is said to be bounded if it is both bounded from below and above

ie $\exists\ M > 0$ such that

$|x| \leq M \quad \forall\ x \in B$

$\Rightarrow -M < x < M \quad \forall\ x \in B$

Example

The set $S = \{-1, 0, 4, 5\}$

the set $S = \{ \cdots -2, -1, 0, 1, 2, 3 \cdots \}$

$\mathbb{N} = \{1\ 2\ 3 \cdots\}$

→ bounded below not above

$S = \{\frac{n}{2n+1},\ n = 1, 2, 3 \cdots\}\ n \in \mathbb{N}$

$2n+1 \geq 2n \quad \forall\ n \in \mathbb{N}$

$\frac{n}{2n+1} \leq \frac{n}{2n} \quad \forall\ n \in \mathbb{R}$

$= \{\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{5}{11}, \cdots\}$

Infimum of a subset $\mathbb{R}$ (Inf S)

Def: If a subset S of $\mathbb{R}$ is bounded below by α, then α is called the lower bound for S. and if S is bounded above by β the β is said to be upper bounded for S.

Def: let S be a subset of $\mathbb{R}$ which is bounded below The greates lower bound for S (Inf S) denoted by α_0 is a real no. that satisfy the following conditions.

1. $\alpha_0 \leq x \quad \forall\ x \in S$
2. if $\alpha \leq x \quad \forall\ x \in S$ then $\alpha \leq \alpha_0 \quad \exists\ x_0 \in S$

$\alpha_0 \leq x_0 < \alpha_0 + \varepsilon$

Theorem 1

let S be a subset of $\mathbb{R}$ which is bounded below

DATE

then $\alpha_0 = \text{Inf} S$ if and only if

(i) α_0 is a lower bound for S

(ii) $\forall \varepsilon > 0 \ \exists x_0 \in S$ such that $\alpha_0 \leq x_0 < \alpha_0 + \varepsilon$

Proof

$\Rightarrow$ let $\alpha_0 = \text{Inf} S$

1. $\alpha_0 \leq x \ \forall x \in S$
2. if α is any other lower bound for S then $\alpha \leq \alpha_0$

let $\varepsilon > 0$ $\alpha_0 + \varepsilon \geq \alpha_0$

$\Rightarrow \alpha_0 + \varepsilon$ is not lower bound for S.

Hence $\exists x_0 \in S$ s.t $\alpha_0 \leq x_0 \leq \alpha_0 + \varepsilon$

$\Rightarrow$

$\Rightarrow$ let (i) (ii) of theorem 1 hold then show that $\alpha_0 = \text{Inf} S$

let be L.b for S we show that $p \leq \alpha_0$

Suprimum of a subset of $\mathbb{R}$

Def: let S be subset of $\mathbb{R}$ which is bounded from above. The least upper bound for S or (suprimum of S) denoted by sup S is a real no. say β_0 that satisfy the following conditions

(i) $x \leq \beta_0 \ \forall x \in S$

(ii) If $x \leq \beta \ \forall x \in S$ then $\beta_0 \leq \beta$

Theorem 2

let S be a subset of $\mathbb{R}$ which is bounded from above then $\beta_0 = \sup S$ if and only if

(i) $x \leq \beta_0 \ \forall x \in S$

(ii) $\forall \varepsilon > 0 \ \exists x_0 \in S$ such that $\beta_0 - \varepsilon < x_0 \leq \beta_0$

Proof

let $\beta_0 = \sup S$

(i) $x \leq \beta_0 \ \forall x \in S \Rightarrow \beta_0$ is an upper bound for S

let $\varepsilon > 0$ $\beta_0 \geq \beta_0 - \varepsilon$ $\beta_0 - \varepsilon$ is not an upper bound for S

$\therefore \exists x_0 \in S$ such that $\beta_0 - \varepsilon < x_0 \leq \beta_0$

Example.

DATE

suppose that

$S = \{ \frac{n}{2n+1}, n = 1, 2, 3 - - - \}$

that $\sup S = \frac{1}{2}$

proof

$\frac{n}{2n+1} \leq \frac{n}{2n} = \frac{1}{2} \quad \forall n \geq 1$

$\frac{1}{2}$ is an upper bounde for S

let $\varepsilon > 0$ be arbitrary

$x_0 = \frac{n_0}{2n_0+1} \quad n_0 \in \mathbb{N}$

$\frac{1}{2} - \varepsilon < x_0 \leq \frac{1}{2}$

$\frac{1}{2} - \varepsilon < \frac{n_0}{2n_0+1} \leq \frac{1}{2}$

$\frac{1}{2} - \varepsilon < \frac{n_0}{2n_0+1}$

$n_0 > \frac{1-2\varepsilon}{4\varepsilon}$

if we choose $n_0 > \frac{1-2\varepsilon}{4\varepsilon}$

then $\frac{1}{2} - \varepsilon < \frac{n_0}{2n_0+1}$ Hence

$\frac{1}{2} = \sup . S$

Show that $\inf S = \frac{1}{3}$

$\frac{1}{3} \leq x \quad \forall x \in S$

$\forall \varepsilon > 0 \ \exists x_0 \in S \ \frac{1}{3} \leq x_0 < \frac{1}{3} + \varepsilon$

(1) $\frac{n}{2n+1} \geq \frac{1}{3} \quad \forall n \geq 1$

let $a_n = \frac{n}{2n+1} \quad a_{n+1} = \frac{n+1}{2(n+1)+1}$

$a_{n+1} - a_n = \frac{n+1}{2(n+1)+1} - \frac{n}{2n+1}$

$a_{n+1} - a_n = \frac{(2n+1)(n+1) - (2n+3)n}{(2n+1)(2(n+1)+1)}$

$= \frac{2n^2 + 2n + n + 1 - 2n^2 - 3n}{(2n+1)(2n+3)}$

$\cancel{a_{n+1} - a_n} = \frac{1}{(2n+1)(2n+3)}$

$a_{n+1} - a_n = \frac{1}{(2n+1)(2n+3)} \geq 0$

$\Rightarrow a_{n+1} \geq a_n \quad \forall n \geq 1$

$a_{n+1} - a_n \geq 0$

$a_{n+1} \geq a_n \quad \forall n \geq 1$

$a_2 \geq a_1$

$a_3 \geq a_2 \Rightarrow a_n = a_1 = \frac{1}{3}$

$a_4 \geq a_3 \quad a_n \geq \frac{1}{3} \quad \forall n \geq 1$

$\frac{n}{2n+1} \geq \frac{1}{3} \quad \forall n \geq 1$

$\Rightarrow \frac{1}{3}$ is lower bounde for S

(11) let $\varepsilon > 0$ be arbitrary.

$\exists x_0 \in S \quad \frac{1}{3} \leq x_0 \leq \frac{1}{3} + \varepsilon$

$\frac{1}{3} \leq \frac{n_0}{2n_0+1} \leq \frac{1}{3} + \varepsilon$ - - - *

if we choose $n_0 = 1$ will hold

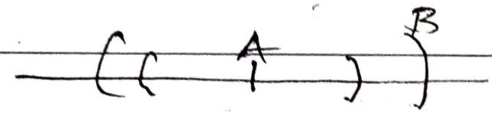
This $\frac{1}{3} = \inf S$.

Basic properties of supremum and infimum

properties 1

let A and B be bounded subset of $\mathbb{R}$ such that $A \subseteq B$. then

$\inf B \leq \inf A \leq \sup A \leq \sup B$



proof $A \in B$

let $\alpha = \inf B$

$\alpha \leq x \quad \forall x \in B$

$\Rightarrow \alpha \leq x \quad \forall x \in A$

$\Rightarrow \alpha$ is a lower bound for A

since α is a lower bound for A then $\alpha \leq \inf A$

but $\alpha = \inf B$

$\Rightarrow \inf B \leq \inf A$ (*)

it is very clear that

$\inf A \leq x \leq \sup A \quad \forall x \in A$

$\Rightarrow \inf A \leq \sup A$ (**)

To show $\sup A \leq \sup B$

let $\beta = \sup B$

$x \leq \beta \quad \forall x \in B$

since $A \subseteq B \Rightarrow x \leq \beta \quad \forall x \in A$

$\Rightarrow \beta$ is a upper bound for A $\beta \geq \sup A$

$\Rightarrow \sup A \leq \sup B$ (***)

$\inf B \leq \inf A \leq \sup A \leq \sup B$

Proposition 2 let S be a subset of $\mathbb{R}$ which is bounded then the following equation holds.

$\sup(-S) = -\inf S$

$\inf(-S) = -\sup S$

$-S = \{-x ; x \in S\}$

proof

let $\alpha = \inf S$

(i) $\alpha \leq x \quad \forall x \in S$

(ii) $\forall \varepsilon > 0 \ \exists x_0 \in S$

$\alpha \leq x_0 < \alpha + \varepsilon$

(i) $-x \leq -\alpha \quad \forall -x \in -S$

$-\alpha$ is an upper bound for $-S$

(ii) $\forall \varepsilon > 0 \ \exists -x_0 \in -S$

$-\alpha - \varepsilon < -x_0 \leq -\alpha$

$\Rightarrow -\alpha = \sup(-S)$

$-\inf S = \sup(-S)$

proposition 3

let $\{a_i\}$ and $\{b_i\}$ $i = 1, 2, 3 \ldots$ be bounded subset of $\mathbb{R}$ then

DATE

(a) $\sup_{i\geq 1}\{a_i+b_i\} \leq \sup_{i\geq 1} a_i + \sup_{i\geq 1} b_i$

(b) $\inf_{i\geq 1}\{a_i+b_i\} \geq \inf_{i\geq 1} a_i + \inf_{i\geq 1} b_i$

proof

(a) let $\alpha = \sup_{i\geq 1} a_i$ $\beta = \sup_{i\geq 1} b_i$

$\alpha \geq a_i \ \forall i \geq 1$ $\beta \geq b_i \ \forall i \geq 1$

$\alpha - \varepsilon < x \leq \alpha \quad \forall x \in \mathbb{R}$

$\beta - \varepsilon < x \leq \beta \quad \forall x \in \mathbb{R}$

$\alpha+\beta$ is an u.b $\{a_i+b_i\} \ \forall i \geq 1$

$\Rightarrow \alpha+\beta \geq \sup(a_i+b_i) \ \forall i \geq 1$

$\sup a_i + \sup b_i \geq \sup(a_i+b_i)$

$\sup(a_i+b_i) \leq \sup a_i + \sup b_i \ \forall i \geq 1$

(b) $\inf\{a_i+b_i\} \geq \inf a_i + \inf b_i$

proof

let $\alpha = \inf a_i$ $\beta = \inf b_i$

$\alpha \leq a_i \ \forall i \geq 1$ $\beta \leq b_i \ \forall i \geq 1$

$\alpha \leq x \leq \alpha + \varepsilon \quad \forall i \geq 1$

$\beta \leq x \leq \beta + \varepsilon \quad \forall i \geq 1$

$\alpha+\beta \leq \inf(a_i+b_i) \quad \forall i \geq 1$

$\inf a_i + \inf b_i \leq \inf(a_i+b_i)$

$\inf(a_i+b_i) \geq \inf a_i + \inf b_i$

proposition 4

let S be a subset of $\mathbb{R}$ which is bounded below such that $\inf S > 0$ let $1/S$ denote the subset of $\mathbb{R}$ defined as $1/S = \{1/x : x \in S\}$ then $\sup(1/S) = \frac{1}{\inf S}$

to proof

~~let~~ $\sup(1/S) = \frac{1}{\inf S}$

let $\alpha = \inf S$

$\Rightarrow \alpha \leq x \ \forall x \in S$

$\frac{1}{x} \leq \frac{1}{\alpha} \ \forall \ \frac{1}{x} \in 1/S$

$\Rightarrow \frac{1}{\alpha}$ is an u.b for $1/S$

$\sup(1/S) \leq \frac{1}{\alpha}$

$\Rightarrow \sup(1/S) \leq \frac{1}{\inf S}$ --- (1)

let $B = \sup(1/S)$

$y \leq B \ \forall y \in 1/S$

$\frac{1}{y} \geq \frac{1}{B} \ \forall \frac{1}{y} \in S$

$\Rightarrow \frac{1}{B}$ is a lower bound for S.

but $\alpha = \inf S$

$\Rightarrow \frac{1}{B} \leq \alpha \Rightarrow B \geq \frac{1}{\alpha}$

$\Rightarrow \sup(1/S) \geq \frac{1}{\inf S}$ --- (2)

Exercise

(1) let A and B be any two bounded non-empty subset of $\mathbb{R}$ and let $A+B = \{x+y : x \in A, y \in B\}$

show that

$\sup(A+B) = \sup(A) + \sup(B)$

DATE 6/2/2018

(2) let S be a subset of $\mathbb{R}$ bounded from above and let $P \subseteq S$ satisfy the following conditions $\forall x \in S \; \exists r \in P$ such that $x \le r$ prove that $\sup S = \sup P$

(3) let S be a bounded subset of $\mathbb{R}$ for $m \in \mathbb{R}$ let $S+m = \{x+m : x \in S\}$ prove that $\sup(S+m) = m + \sup S$

Solution

(1) let A and B be any two bounded nonempty subset of $\mathbb{R}$ and let $A+B = \{x+y : x \in A, y \in B\}$ show that $\sup(A+B) = \sup A + \sup B$

proof

let $x = \sup A$

$y = \sup B$

$x \le A \quad \forall x \in A$

$y \le B \quad \forall y \in B$

$A - \varepsilon \le x \le A$

$B - \varepsilon \le y \le B$

$(A+B) - \varepsilon \le x+y \le (A+B)$

$\sup(A+B) = \sup A + \sup B$

Lemma: let we a fixed real number suppose for any $\varepsilon > 0$ we have $|c| < \varepsilon$ then $c = 0$

proof

If $c \ne 0$ then $\frac{1}{2}|c| > 0$

If we let $\varepsilon = \frac{1}{2}|c| > 0$

Then by the hypothesis is

$\Rightarrow |c| < \frac{1}{2}|c|$

this is contradiction therefore $c = 0$

Corollary: let c we a fixed real number: suppose that for some $K > 0$ and every $\varepsilon > 0$ we have $|c| < K\varepsilon$ then $c = 0$

A function denoted by $[x]$ takes a real numbers x input and output the nearest integer which is equal or less than the number x

$f : \mathbb{R} \to \mathbb{N}$

$[x] = n$

let x be an arbitrary real number ~~then~~ there is exactly one integer n which satisfies the inequality $n \le x < n+1$. This n is called the greatest integer in and is denoted by $[x]$

e.g. $x = -0.01$

$[x] = [-0.01] = -1 \le -0.01 \le -1+1$

$[-0.01] = -1$

$[0.8] = 0 \quad 0 \le 0.8 \le 0+1$

$[-0.2] = -1 \quad -1 \le -0.2 \le -1+1$

$[7.5] = 7 \quad 7 \le 7.5 \le 7+1$

$[1/2] = 0 \quad 0 \le 1/2 \le 0+1$

Theorem let $\mathbb{R}$ be a real number system then if x is any element of $\mathbb{R}$ there is an integer n such that $n > x$

SEQUENCE of $\mathbb{R}$

Def A sequence $\{x_n\}_{n=1}^{\infty}$ or $(x_n)_{n=1}^{\infty}$ or $\{x_n\}$ of real number is a function f of the natural number in to the set of $\mathbb{R}$

$f : \mathbb{N} \to \mathbb{R}$ ~~f(x)~~

$f(n) = x_n$

e.g.

$(1/n)_{n=1}^{\infty} = \{1/1, 1/2, 1/3, 1/4, \ldots, \frac{1}{n-1}, \frac{1}{n+1}, \frac{1}{n+2} \ldots\}$

$\{6\}_{n=1}^{\infty} = \{6\ 6\ 6\ 6\}$

$\{(-1)^n\}_{n=1}^{\infty} = \{-1, 1, -1, 1, -1 \ldots\}$

$\{\frac{n}{n+1}\}_{n=1}^{\infty} = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5} \ldots\}$

$\{\frac{1}{n}\} = \{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots \frac{1}{n}\}$

Def A sequence $\{x_n\}_{n=1}^{\infty}$ of real number is said to converge to a number L if and only if for such $\varepsilon > 0$ there exist a natural number $n(\varepsilon)$ such that

$|x_n - L| < \varepsilon \quad \forall n \ge n(\varepsilon)$

In this case we write

$\lim_{x \to \infty} x_n = L$ or $x_n \xrightarrow[n \to \infty]{} L$ or

$x_n \to L$ as $n \to \infty$

$\lim x_n = L$

A sequence which does not converge to some number is said to diverge

$x_n \not\to L$

$x \le x < x+1$

DATE

$x_1, x_2, x_3 \dots x_{n-1}, x_{n+1} \dots x_{n(\varepsilon)}$

$x_{n(\varepsilon)+1}$ $x_{n(\varepsilon)+2} \dots$ will satisfy

Example

$\{\frac{1}{n}\}_{n=1}^{\infty}$ show that set converges $=0$ let $\varepsilon = 1$

proof

$|\frac{1}{n} - 0| < \varepsilon$

$|\frac{1}{1} - 0| < 1 = |1-0| < 1$ $\quad 1<1$ ✗

$n=2$

$|\frac{1}{2} - 0| < 1 = \frac{1}{2} < 1$

$|\frac{1}{3} - 0| < 1 = \frac{1}{3} < 1$

$|\frac{1}{4} - 0| < 1 = \frac{1}{4} < 1$

$n(\varepsilon) = 2$

if $\varepsilon = 1$ $\forall n \geq 2$

① let $\varepsilon > 0$ be given we produce $n(\varepsilon)$: $|\frac{1}{n} - 1| < \varepsilon$ $\forall n \geq n(\varepsilon)$

$|\frac{1}{n} - 0| = \frac{1}{n} < \varepsilon \Rightarrow n > \frac{1}{\varepsilon}$

choose $n(\varepsilon) = [\frac{1}{\varepsilon}] + 1$ so that $|\frac{1}{n} - 0| < \varepsilon$ $\forall n \geq n(\varepsilon)$

② prove that the sequence $\left(\frac{n}{n+1}\right)_{n=1}^{\infty}$ converges to 1

proof

let $\varepsilon > 0$ be given we produce $n(\varepsilon)$ such that $|\frac{n}{n+1} - 1| < \varepsilon$ $\forall n \geq n(\varepsilon)$

$|\frac{n}{n+1} - 1| = |\frac{-1}{n+1}| = \frac{1}{n+1} < \frac{1}{n}$ $\forall n \geq 1$

$\Rightarrow |\frac{n}{n+1} - 1| < \frac{1}{n} < \varepsilon \Rightarrow n > \frac{1}{\varepsilon}$

if we choose $n(\varepsilon) = [\frac{1}{\varepsilon}] + 1$ then inequality

$|\frac{n}{n+1} - 1| < \varepsilon$ $\forall n \geq n(\varepsilon)$ is satisfied

③ prove that $\left|\frac{1}{n^2}\right|_{n=1}^{\infty}$ converges to 0

proof

let $\varepsilon > 0$ be given we construct $n(\varepsilon)$ so that $|\frac{1}{n^2} - 0| < \varepsilon$ $\forall n \geq n(\varepsilon)$

~~$|\frac{1}{n^2} - 0| \geq \frac{1}{n} < \varepsilon$~~

$|\frac{1}{n^2} - 0| = \frac{1}{n^2} < \frac{1}{n}$ $\forall n \geq 1$

$\Rightarrow |\frac{1}{n^2} - 0| < \frac{1}{n} < \varepsilon \Rightarrow n \geq \frac{1}{\varepsilon}$

choose $n(\varepsilon) = (\frac{1}{\varepsilon}) + 1$ so that $|\frac{1}{n^2} - 0|$ $\forall n \geq n(\varepsilon)$

④ prove that $\lim_{x \to \infty} \frac{3n+1}{7n-4} = \frac{3}{7}$

$a_n = \left(\frac{3n+1}{7n-4}\right) \lim_{x \to \infty} \to \frac{3}{7}$

proof

let $\varepsilon > 0$ be given we produce $n(\varepsilon)$ such that $|a_n - \frac{3}{7}| < \varepsilon$ $\forall n \geq n(\varepsilon)$

$\left|\frac{3n+1}{7n-4} - \frac{3}{7}\right| = \left|\frac{7(3n+1) - 3(7n-4)}{7(7n-4)}\right|$

$= \left|\frac{21n + 7 - 21n + 12}{7(7n-4)}\right|$

DATE

$$\left|\frac{19}{7(7n-4)}\right|$$

if $\frac{7n}{2} > 4 \Rightarrow n > 8/7$

then

$$\left|\frac{3n+1}{7n-4} - \frac{3}{7}\right| = \frac{19}{7(7n-4)} < \frac{19}{7(7n-\frac{7n}{2})}$$

$\forall n > 8/7$

$$\left|\frac{3n+1}{7n-4} - \frac{3}{7}\right| < \frac{19}{7(\frac{7n}{2})} = \frac{38}{49n}$$

$\forall n > 8/7 \left|\frac{3n+1}{7n-4} - \frac{3}{7}\right| < \frac{38}{49n} < \varepsilon$

$\Rightarrow n > \frac{38}{49\varepsilon}$

we choose

$n(\varepsilon) = \max\left[\left(\frac{38}{49\varepsilon}\right)+1, \left[\frac{8}{7}\right]+1\right]$

~~so that~~ ~~$\forall n \geq n(\varepsilon)$~~

so that $\left|\frac{3n+1}{7n-4} - \frac{3}{7}\right| < \varepsilon \ \forall n \geq n(\varepsilon)$

(5) prove that $\lim_{n\to\infty} \frac{3n+1}{7n-45} = 3/7$

Proof

let $\varepsilon > 0$ be given we produce $n(\varepsilon)$ so that

$$\left|\frac{3n+1}{7n-45} - \frac{3}{7}\right| < \varepsilon \quad \forall n \geq n(\varepsilon)$$

$$\left|\frac{3n+1}{(7n-45)} - \frac{3}{7}\right| = \left|\frac{7(3n+1) - 3(7n-45)}{7(7n-45)}\right|$$

x_n ~~$\frac{4n-7n}{2}$~~

$$\left|\frac{21n+7-21n+135}{7(7n-45)}\right|$$

$$\left|\frac{142}{7(7n-45)}\right|$$

if $\frac{7n}{2} > 45$ then $n > \frac{90}{7}$

$$\left|\frac{3n+1}{7n-45}\right| < \frac{142}{7(7n-\frac{7n}{2})} = \frac{142}{7(\frac{7n}{2})}$$

if $\frac{284}{49n} < \varepsilon$

$\therefore$ we choose

$n(\varepsilon) = \max\left\{\left[\frac{90}{7}\right]+1, \left(\frac{284}{49\varepsilon}\right)\right\}$

$\forall n > \frac{284}{49\varepsilon}$

Exercise

(6) prove that $\lim_{n\to\infty} \frac{4n^3+3n}{n^3-6} = 4$

theorem: Every convergent sequence is bounded.

Proof → Exercise

Exercise

(1) for each of the following determines whether it converges if it converges guess its limits and prove your guess using $N-\varepsilon$ argument

(a) $a_n = \{\sqrt{n^2+1} - n\}$
(b) $a_n = \{\sqrt{4n^2+n} - 2n\}$
(c) $a_n = \frac{(-1)^n}{n}$

(d) $a_n = \frac{2n-1}{3n+2}$

8/2/2018

Theorem: Every convergent sequence is bounded.

~~proof~~ proof

Let $\{a_n\}_{n=1}^{\infty}$ be a convergent sequence we show that $\{a_n\}$ is bounded.
ie $|a_n| \leq M$ $\forall n \in \mathbb{N}$ where $M > 0$ $M \in \mathbb{R}$

Since a_n is convergent let L be its limit
$\therefore \forall \varepsilon > 0$ $\exists n(\varepsilon) \in \mathbb{N}$ such that
$|a_n - L| < \varepsilon$ $\forall n \geq n(\varepsilon)$ but
$|a_n| - |L| \leq ||a_n| - |L|| \leq |a_n - L| < \varepsilon$
$\forall n \geq n(\varepsilon)$
$\Rightarrow |a_n| - |L| < \varepsilon$ $\forall n \geq n(\varepsilon)$
Let $\varepsilon = 1$.
$\Rightarrow |a_n| < |L| + 1$ $\forall n \geq n(\varepsilon)$

Example
The converse of this theorem is false ie we can have some sequence that are bounded but not convergent.
eg $\{a_n\} = \{(-1)^n\}_{n=1}^{\infty} = \{-1, 1, -1, 1 - \ldots\}$
$\Rightarrow |a_n| = 1$ $\forall n \in \mathbb{N}$ implies that the sequence is bounded but we claim that it is not bounded.
To prove we claim suppose that the sequence $\{a_n\}$ is convergent to L
ie $\forall \varepsilon > 0$ $\exists n(\varepsilon)$; $|a_n - L| < \varepsilon$ $\forall n \geq n(\varepsilon)$
$\Rightarrow |(-1)^n - L| < \varepsilon$ $\forall n \geq n(\varepsilon)$
Implication
$|(-1)^{n(\varepsilon)+1} - L| < \varepsilon$ and
$|(-1)^{n(\varepsilon)+2} - L| < \varepsilon$
let $\varepsilon = 1/2$
$|(-1)^{n(\varepsilon)+1} - L| < 1/2$ and
$|(-1)^{n(\varepsilon)+2} - L| < 1/2$
but
$2 = |(-1)^{n(\varepsilon)+1} - (-1)^{n(\varepsilon)+2}| =$
$= |(-1)^{n(\varepsilon)+1} - L + L - (-1)^{n(\varepsilon)+2}|$

$2 = |(-1)^{n(\varepsilon)+1} - L + L - (-1)^{n(\varepsilon)+2}| \leq |(-1)^{n(\varepsilon)+1} - L| + |L - (-1)^{n(\varepsilon)+2}| \leq 1/2 + 1/2$

$\Rightarrow 2 \leq 1/2 + 1/2$

$\Rightarrow 2 \leq 1$ * $\Rightarrow \{a_n\}$ is not convergen

<u>Theorem</u> suppose that $\{a_n\}$ and $\{b_n\}$ are convergent sequence and that $a_n \to a$ and $b_n \to b$ if $a_n \leq b_n$ $\forall n \geq 1$ ~~and~~ ~~$b_n \to b$~~ then $a \leq b$

<u>proof</u>: suppose $a > b$

let $\varepsilon = \frac{a-b}{2} > 0$ $\quad -\varepsilon < a_n - a < \varepsilon$

$a_n \to a \Rightarrow \forall \varepsilon > 0 \ \exists n_1(\varepsilon) : |a_n - a| < \varepsilon \ \forall n \geq n(\varepsilon)$

$b_n \to b \Rightarrow \forall \varepsilon > 0 \ \exists n_2(\varepsilon) : |b_n - b| < \varepsilon \ \forall n \geq n(\varepsilon)$

let $n_1(\varepsilon) = N_1$

$n_2(\varepsilon) = N_2$

$a - \frac{(a-b)}{2} < a_n < \frac{(a-b)}{2} + a$ ----- *

$\boxed{\frac{a+b}{2} < a_n} < \frac{3a-b}{2}$ ----- * 1

$b - \frac{(a-b)}{2} < b_n < \frac{(a-b)}{2} + b$

$\frac{3b-a}{2} < \boxed{b_n < \frac{a+b}{2}}$ ----- 2

1 and 2 it clear that $b_n \leq \frac{a+b}{2} \leq a_n$

$\Rightarrow b_n \leq a_n$ *

$\Rightarrow a \leq b$

DATE

* $\{\sqrt{n^2+1} - n\}$.

let $a_n = \sqrt{n^2+1} - n$ so that

$\Rightarrow a_n + n = \sqrt{n^2+1}$

$(a_n+n)^2 = n^2+1$

$a_n^2 + 2na_n + \cancel{n^2} = \cancel{n^2}+1$

$a_n^2 + 2n a_n = 1$

$\Rightarrow 2n a_n \leq 1$

since $a_n^2 \geq 0 \ \forall n$

or

$\sqrt{n^2+1} - n = \frac{1}{\sqrt{n^2+1}+n}$

$\frac{(\sqrt{n^2+1} - n)(\sqrt{n^2+1}+n)}{\sqrt{n^2+1}+n} = \frac{n^2+1-n^2}{\sqrt{n^2+1}+n}$

$= \frac{1}{\sqrt{n^2+1}+n}$

$\left|\frac{1}{\sqrt{n^2+1}+n} - 0\right| = \left|\frac{1}{\sqrt{n^2+1}+n}\right| = \frac{1}{\sqrt{n^2+1}+n}$

$\frac{1}{\sqrt{n^2+1}+n} < \frac{1}{n+n} = \frac{1}{2n}$

$|a_n - 0| \leq \frac{1}{2} < \varepsilon \quad n > \frac{1}{2\varepsilon}$

$|a+b| \le |a|+|b|$ — triangular inequality

Theorem DATE

~~$a_n \to a$~~

Let $\{a_n\}, \{b_n\}$ be sequence of $\mathbb{R}$ and let $\alpha \in \mathbb{R}$

suppose $a_n \to a$ and $b_n \to b$

Then

(a) $a_n + b_n \longrightarrow a+b$ (b) $a_n b_n \longrightarrow ab$

(c) $\alpha + a_n \longrightarrow \alpha + a$, $\alpha a_n \longrightarrow \alpha a$

(d) $\frac{1}{b_n} \longrightarrow \frac{1}{b}$ and $\frac{a_n}{b_n} \longrightarrow \frac{a}{b}$ provided $b_n \neq 0 \ \forall n \ge 1$ $b \neq 0$

Proof

$a_n \to a \Rightarrow \forall \varepsilon > 0 \ \exists n_1$ s.t $|a_n - a| < \frac{\varepsilon}{2} \ \forall n \ge n_1$

$b_n \to b \Rightarrow \forall \varepsilon > 0 \ \exists n_2$ s.t $|b_n - b| < \frac{\varepsilon}{2} \ \forall n \ge n_2$

let $m = \max\{n_1, n_2\}$

$|a_n + b_n - (a+b)| \le |a_n - a| + |b_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \ \forall n \ge m$

$\Rightarrow |a_n + b_n - (a+b)| < \varepsilon \ \forall n \ge m.$ $a_n + b_n \to a+b$

$a_n \to a$ $b_n \to b$ $a_n b_n \longrightarrow ab$

$\exists M: |a_n| < M$ ~~$\forall n \ge 1$~~ $\forall \varepsilon > 0$ $\frac{\varepsilon}{2|b|} > 0$

$\forall n \ge 1$ $|a_n - a| < \frac{\varepsilon}{2|b|} \ \forall n \ge n_1$

We show

$|a_n b_n - ab| < \varepsilon \ \forall n \ge n_2$

$a_n b_n - ab = a_n(b_n - b) + b(a_n - a)$

$b_n \to b \Rightarrow |b_n - b| < \frac{\varepsilon}{2M} \ \forall n \ge n_2$

Now

let $m = \max\{n_1, n_2\}$

$|a_n b_n - ab| = |a_n(b_n - b) + b(a_n - a)| \le |a_n||b_n - b| + |b||a_n - a|$

$\Rightarrow |a_n b_n - ab| \le |a_n||b_n - b| + |b||a_n - a|$

$\le M \frac{\varepsilon}{2M} + |b| \frac{\varepsilon}{2|b|} \quad \forall n \ge m$

$\Rightarrow |a_n b_n - ab| < \varepsilon \quad \forall n \ge m$

Theorem if $a_n \to a$ then $|a_n| \to |a|$

$a_n \to a$

let $\varepsilon > 0$ be given then $\exists n(\varepsilon)$; $|a_n - a| < \varepsilon \ \forall n \geq n(\varepsilon)$

but

$||a_n| - |a|| \leq |a_n - a| < \varepsilon \quad \forall n \geq n(\varepsilon)$

$\Rightarrow ||a_n| - |a|| < \varepsilon \quad \forall n \geq n(\varepsilon)$

~~Theorem~~

Theorem let $\{a_n\}$ be convergent sequence then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$

let a be the limit of the sequence a_n

$\Rightarrow \forall \varepsilon > 0 \ \exists n(\varepsilon)$ s.t $|a_n - a| < \varepsilon \quad \forall n \geq n(\varepsilon)$

but $n+1 > n > n(\varepsilon)$

$\Rightarrow |a_{n+1} - a| < \varepsilon \quad \forall n \geq n(\varepsilon)$

$\Rightarrow a_{n+1} \rightarrow a.$

MONOTONE SEQUENCES

A sequence $\{a_n\}$ is said to be monotone Increasing sequence if $a_{n+1} \geq a_n \ \forall n \geq 1$. And it is called strictly Increasing if $a_{n+1} > a_n$. $\forall n \geq 1$

A sequence is said to be ~~moto~~ monotone decreasing sequence if $a_{n+1} \leq a_n \ \forall n \geq 1$ And is called strictly decreasing if $a_{n+1} < a_n \ \forall n \geq 1$.

- To check if the sequence is monotone we use the following methods.

- difference if $a_{n+1} - a_n \geq 0$ then a_n is M.I

$a_{n+1} - a_n \leq 0$ then a_n is M.D

$\frac{a_{n+1}}{a_n} \geq 1$ a_n is M.I

$\frac{a_{n+1}}{a_n} \leq 1$ a_n is M.D

- $a_n = f(x),\ x \geq 1$ if $f'(x) > 0$ a_n is M.I
  if $f'(x) < 0$ a_n is M.D
- by induction $a_{n+1} \leq a_n\ \forall n \geq 1$ a_n is M.D
  $a_{n+1} \geq a_n\ \forall n \geq 1$ a_n is M.I

Examples: $\{a_n\}_{n=1}^{\infty} = \{\frac{1}{n}\}_{n=1}^{\infty}$, $a_n = \frac{1}{n}$; $a_{n+1} = \frac{1}{n+1}$

$a_{n+1} - a_n = \frac{1}{n+1} - \frac{1}{n} = \frac{n-(n+1)}{n(n+1)} = \frac{-1}{n(n+1)} < 0$

$a_{n+1} - a_n < 0\ \forall n \geq 1$

a_n is M.D sequence

$\{a_n\}_{n=1}^{\infty} = \{n^2\}_{n=1}^{\infty}$, $a_n = n^2$, $a_{n+1} = (n+1)^2$

$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{n^2} = \frac{n^2+2n+1}{n^2} > 1$

$a_{n+1} - a_n = (n+1)^2 - n^2 = 2n+1 > 0$ M.I

$a_n = \{(-1)^n\}_{n=1}^{\infty}$ $a_{n+1} = (-1)^{n+1}$

$a_n = (-1)^n$

$a_{n+1} - a_n = (-1)^{n+1} - (-1)^n = \begin{cases} -2 & \text{if } n \text{ even} \\ 2 & \text{if } n \text{ odd} \end{cases}$

Theorem

a. A M.I sequence which is bounded above convergen
b. A M.D sequence which is bounded below convergens

proof

let $\{a_n\}$ be a M.I sequence which is odd from above

$\Rightarrow$ sup a_n exist let $\beta = \sup_{n \geq 1} a_n$

$\Rightarrow$ (i) $a_n \leq \beta\ \forall n \geq 1$

(ii) $\forall \varepsilon > 0\ \exists\, a_{n_0} \in \{a_n\}$ s.t $\beta - \varepsilon < a_{n_0} \leq \beta$

Since a_n is M.I then $\forall n \geq n_0$ $a_n \geq a_{n_0}$ ..

DATE

for (ii) $\beta-\varepsilon < a_{n_0} \leq a_n \leq \beta < \beta+\varepsilon \quad \forall n \geq n_0$

$\Rightarrow \beta-\varepsilon < a_n < \beta+\varepsilon \quad \forall n \geq n_0$

$|a_n - \beta| < \varepsilon \quad \forall n \geq n_0$

$\Rightarrow a_n \to \beta$

20/2/2018

$a_n = \left(1+\frac{1}{n}\right)^n \quad b_n = \left(1+\frac{1}{n}\right)^{n+1}$

claim

① $a_n \leq b_n \quad \forall n \geq 1$

② a_n is M.I

③ b_n is M.D

④ a_n and b_n convergencs (to the same limit) $\to$ (exercise)

proof

① $a_n \leq b_n \quad \forall n \geq 1$

$b_n = \left(1+\frac{1}{n}\right)^{n+1} = \left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right)$

$b_n = a_n\left(1+\frac{1}{n}\right)$

$b_n \geq a_n \quad \forall n \geq 1$ since $\left(1+\frac{1}{n}\right) \geq 1$

Bernoulle Inequality — Lemma

let $p > -1$ $p \neq 0$ then $\forall n \geq 2$ we have $(1+p)^n > 1+np$

$P(n) = (1+p)^n$

$P(2) = (1+p)^2 = 1+2p+p^2 > 1+2p \quad p^2 > 0$

$P(2)$ is true

suppose $P(k)$ is true i.e $(1+p)^k > 1+kp$

we show $P(k+1)$ is also true

ie $P(k+1) = (1+p)^{k+1} = (1+p)^k \cdot (1+p) > (1+kp)(1+p)$ $\quad 1+p+kp+kp^2$

$P(k+1) > 1+(k+1)p+kp^2 > 1+(k+1)p \quad kp^2 \geq 0$

$\therefore P(k+1) = (1+p)^{k+1} > 1+(k+1)p$

$\therefore P(n)$ is true $\forall n \geq 2$

$$\frac{n^2}{(n+1)^2(n-1)} < \frac{1}{n+1}$$

DATE

$$\frac{a_n}{a_{n-1}} = \frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{1}{n+1}\right)^{n-1}} = \frac{\left(\frac{n+1}{n}\right)^n}{\left(\frac{n}{n-1}\right)^n\left(\frac{n}{n-1}\right)^{-1}} = \left(\frac{n+1}{n}\right)^n\left(\frac{n-1}{n}\right)^n\left(\frac{n}{n-1}\right)$$

By bernoulli inequality

$$\left(\frac{n^2-1}{n}\right)^n\left(\frac{n}{n-1}\right) = \left(1-\frac{1}{n^2}\right)^n\left(\frac{n}{n-1}\right) > \left(1-\frac{n}{n^2}\right)\left(\frac{n}{n-1}\right)$$

$$\frac{a_n}{a_{n-1}} > \left(1-\frac{1}{n}\right)\left(\frac{n}{n-1}\right) = \frac{n-1}{n}\cdot\frac{n}{n-1} = 1$$

$\Rightarrow \frac{a_n}{a_{n-1}} > 1$ $\therefore$ a_n is M.I

b_n is M.D ??

$$b_n = \left(1+\frac{1}{n}\right)^{n+1}$$

$$\frac{b_{n-1}}{b_n} = \frac{\left(1+\frac{1}{n-1}\right)^n}{\left(1+\frac{1}{n}\right)^{n+1}} = \frac{\left(\frac{n}{n-1}\right)^n}{\left(\frac{n+1}{n}\right)^n\left(\frac{n+1}{n}\right)} = \left(\frac{n}{n-1}\cdot\frac{n}{n+1}\right)^n\left(\frac{n}{n+1}\right)$$

$$= \left(\frac{n^2}{n^2-1}\right)^n\left(\frac{n}{n+1}\right)$$

$$\begin{array}{r|l} \frac{1}{n^2-1} & n^2 \\ & n^2-1 \\ \hline & 1 \end{array}$$

$$\frac{n^2}{n^2-1} = 1+\frac{1}{n^2-1}$$

$$\frac{b_{n-1}}{b_n} = \left(1+\frac{1}{(n+1)(n-1)}\right)^n\left(\frac{n}{n+1}\right) > \left(1+\frac{n}{(n+1)(n-1)}\right)\left(\frac{n}{n+1}\right)$$

$$\frac{b_{n-1}}{b_n} > \frac{n}{n+1} + \frac{n^2}{(n+1)^2(n-1)} > \frac{n}{n+1} + \frac{1}{n+1} = 1$$

a_n is M.I $\Rightarrow a_1 \le a_n \;\forall n \ge 1$

since $a_n \le b_n \;\forall n$ and b_n is M.D $b_n \le b_1 \;\forall n \ge 1$

$a_1 \le a_n \le b_n \le b_1$

DATE

Sequence defined by recurrance relation

Example, Let $b_1 = 1$ and $b_{n+1} = \sqrt{2+b_n}$. show that $\{b_n\}$ is convergent and find its limit

$b_{n+2} = \sqrt{2+b_{n+1}}$

solution

$b_1 = 1$ $b_2 = \sqrt{2+b_1} = \sqrt{2+1} = \sqrt{3}$

$b_3 = \sqrt{2+b_2} = \sqrt{2+\sqrt{3}}$ b_4 - -

Claim b_n is M.I

i.e $b_n \le b_{n+1}$ $\forall n$

$b_1 = 1$, $b_2 = \sqrt{3}$

$1 \le \sqrt{3}$ true for $n=1$

suppose it is true for $n=k$ ie $b_k \le b_{k+1}$ ie $b_k \le \sqrt{2+b_k}$

$b_{k+1} \le b_{k+2}$

$b_{k+1} = \sqrt{2+b_k} \le \sqrt{2+b_{k+1}} = b_{k+2}$

$b_{k+1} \le b_{k+2}$ $\therefore$ it's true for $n = k+1$

Hence $b_n \le b_{n+1}$ $\forall n \ge 1$

$b_1 = 1 \le 2$ so it's true for $n \ge 1$

suppose its true for $n=k$ $b_k \le 2$

$b_{k+1} = \sqrt{2+b_k} \le \sqrt{2+2} = \sqrt{4} = 2$

$b_{k+1} \le 2$

$\therefore$ by mathematical Induction $b_n \le 2$ $\forall n \ge 1$

b_n is convergent

To find the limit

let $x = \lim_{n\to\infty} b_n$ but we now $\lim_{n\to\infty} b_n = \lim_{n\to\infty} b_{n+1}$

$\therefore x = \lim_{n\to\infty} b_{n+1} = \lim_{n\to\infty} \sqrt{2+b_n} = \sqrt{2+x}$

$\Rightarrow x = \sqrt{2+x} \Rightarrow x^2 - 2 - x = 0$

$x =$ or $x =$

$|a| = ? > 0$
$|a| > 0$

DATE

(2) Let $a_1 > 1$ and $a_{n+1} = 2 - \frac{1}{a_n}$ for $n \geq 1$. show that a_n is monotone and bounded.

Solution

$a_1 > 1 \quad a_{n+1} = 2 - \frac{1}{a_n}$

$a_2 = 2 - \frac{1}{a_1}$

$a_{n+1} - a_n = \left(2 - \frac{1}{a_n}\right) - a_n$

$= \frac{2a_n - 1}{a_n} - a_n$

$= \frac{2a_n - 1 - a_n^2}{a_n} = \frac{-(a_n^2 - 2a_n + 1)}{a_n}$

$= \frac{-(a_n - 1)^2}{a_n} < 0 \quad \therefore a_n$ is M.D

claim $a_n > 1 \ \forall n$

for $n = 1 \quad a_1 > 1$ true for $n = 1$

suppose it's for $n = k$ i.e $a_k > 1$

$a_{k+1} = 2 - \frac{1}{a_k}$ since $a_k > 1$, $\frac{1}{a_k} < 1 \qquad a_{k+1} = 2 - \frac{1}{a_k} > 1$

2 − less than 1 > 1

$\therefore a_{k+1} > 1$

Hence by induction $a_n > 1 \ \forall n \geq 1$ and therefore a_n is convergent

let $x = \lim_{n \to \infty} a_n \qquad x = \lim a_n = \lim a_{n+1}$

$\Rightarrow x = \lim_{n \to \infty} 2 - \frac{1}{a_n} = 2 - \frac{1}{x}$

$x = 2 - \frac{1}{x} \Rightarrow x^2 - 2x + 1 = 0 = (x-1)^2$

Establish the convergence or the divergence of the sequence $\{a_n\}$ where $a_n = \left\{\sum_{i=1}^{n} \frac{1}{n+i}\right\}_{n=1}^{\infty}$

$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{n+n}$

$a_1 = \frac{1}{1+1} = 1/2$

$a_2 = \frac{1}{1+1} + \frac{1}{2+2} =$

$a_{n+1} = \frac{1}{(n+1)+1} + \frac{1}{(n+1)+2} + \frac{1}{(n+1)+3} + \cdots + \frac{1}{(n+1)+n} + \frac{1}{(n+1)+(n+1)}$

$a_{n+1} - a_n = \left(\frac{1}{n+2} + \cancel{\frac{1}{n+3}} + \frac{1}{n+4} + \cdots + \cancel{\frac{1}{2n}} + \frac{1}{2n+1} + \frac{1}{2(n+1)} \right) - \left(\frac{1}{n+1} + \cancel{\frac{1}{n+2}} + \frac{1}{n+3} + \cdots \right.$
$\left. \cdots + \frac{1}{2n-1} + 1/2n \right)$

$a_{n+1} - a_n = \frac{1}{2n+1} + \frac{1}{2(n+1)} - \frac{1}{n+1} = \frac{2n+2+2n+1-2(2n+1)}{(2n+1)(2(n+1))}$

$= \frac{1}{(2n+1)(2n+2)} > 0$

$\therefore a_n$ is M.I sequence.

$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{n+n} < \underbrace{\frac{1}{n} + \frac{1}{n} + \frac{1}{n} + \cdots \frac{1}{n}}_{n \text{ times}}$

$\frac{n}{n} = 1$

$a_n < 1 \quad \forall n \geq 1$

Sandwhich theorem 6/3/2018

<u>Theorem</u> suppose $\{a_n\}$ and $\{b_n\}$ are sequences of real numbers. such that for some $N_0 \geq 1$ we have $a \leq b_n \leq a_n \quad \forall n \geq N_0$

If a_n convergence to a then b_n also converges to a

<u>proof</u>: $a \leq b_n \leq a_n \quad \forall n \geq N_0$

suppose $a_n \to a \Rightarrow \forall \varepsilon > 0$ given $\exists n(\varepsilon)$ s.t

$|a_n - a| < \varepsilon \quad \forall n \geq n(\varepsilon)$

DATE

$0 \leq b_n - a$ since $b_n \leq a_n$

$0 < |b_n - a| \leq |a_n - a| < \varepsilon \quad \forall n \geq n(\varepsilon)$

$n = \max\{N_0, n(\varepsilon)\}$

$\Rightarrow |b_n - a| < \varepsilon \quad \forall n \geq n(\varepsilon)$

Examples

① If $|a| < 1$ then $\{a^n\}$ an $\{n a^n\}$ are null sequence

If $a = 0 \Rightarrow a^n = 0$ then $a^n \to 0$

$|a| < 1 \Rightarrow \frac{1}{|a|} > 1 \Rightarrow \frac{1}{|a|} = 1 + x$ for some $x > 0$

$\frac{1}{|a|^n} = (1+x)^n > 1 + nx > nx$

$\frac{1}{|a|^n} > nx \Rightarrow 0 < |a|^n < \frac{1}{nx} \to 0$

$\therefore$ by S.T $|a|^n \to 0$

② show that the sequence $\{a_n\} = \frac{n+3}{n^2-1}$ is a null sequence

Proof

$0 \leq \frac{n+3}{n^2-1} < \frac{n+3n}{n^2-1} < \frac{4n}{n^2 - \frac{n^2}{2}} = \frac{8n}{n^2} = \frac{8}{n}$ $\quad n \geq 2 \quad \frac{n^2}{2} \geq 1 \quad n \geq \sqrt{2}$

$0 < a_n < \frac{8}{n} \quad \forall n \geq 2$

$\frac{8}{n} \xrightarrow{n \to \infty} 0$

$a_n \to 0$ By S.T

③ Show

$\lim_{n \to \infty} \sqrt[n]{n} = 1$

Solution

$a_n = (\sqrt[n]{n} - 1)$ we show $a_n \to 0$

$1 + a_n = n^{\frac{1}{n}} \Rightarrow n = (1 + a_n)^n = 1 + n a_n + \frac{n(n-1)}{2!} a_n^2 + \dots$

$\sqrt[n]{n} \geq 1 \ \forall n$

DATE

$n > \frac{n(n-1)\,a_n^2}{2}$

$\frac{n}{n(n-1)} > \frac{n(n-1)\,a^2}{n}$

$a_n^2 < \frac{2}{n-1}$

$0 \leq a_n < \sqrt{\frac{2}{n-1}} \xrightarrow{n\to\infty} 0$

$\downarrow n\to\infty$

0 by S.T

Exercise

① $|n a^n| \longrightarrow 0$

Let $a_n = \{(-\frac{5}{9})^n\}$ show that a_n converges to 0

let $a = -\frac{5}{9} \Rightarrow |a| = \frac{5}{9} < 1$

$\therefore$ by problem (i) we have seen that $|a| < 1$

$\Rightarrow a^n \longrightarrow 0$

Hence $a_n \longrightarrow 0$

Exercise

① prove that the following sequence converges to 0

(a) $\frac{3n+2}{n^2+2}$ (b) $\frac{n^2+4}{n^3-12}$ (c) $\frac{(-1)^n}{\sqrt{n}}$ (d) $\frac{n^3+2n^2-1}{n^4-n^2+2}$

② prove that the following sequence converges

(a) $\left\{\frac{n-1}{n+1}\right\}$ (b) $\frac{2n^2+1}{n^2+3n}$ (c) $\frac{3n^2-1}{n^2-5n}$, $n \geq 6$

(e) $\{\sqrt{n}(\sqrt{n+1} - \sqrt{n})\}$

③ Suppose that $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences such that $a_n \leq b_n \leq c_n$ $\forall n \in \mathbb{N}$ and that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = a$ prove that $\lim_{n\to\infty} b_n = a$.

$c \in \mathbb{R}$ $|c| < \varepsilon$ $c = 0$

DATE

Uniqueness of limits:

Theorem: If a sequence $\{a_n\}$ converges to a limit a^* then the limit is unique.

proof

by contradiction let a_n converges to two limits a and b with $a \neq b$

$a_n \to a \Rightarrow \forall \varepsilon > 0 \; \exists n(\varepsilon) \in \mathbb{N} : |a_n - a| < \frac{\varepsilon}{2} \; \forall n \geq n(\varepsilon)$

$a_n \to b \Rightarrow \exists n_1(\varepsilon) \in \mathbb{N}$ s.t $|a_n - b| < \frac{\varepsilon}{2} \; \forall n \geq n(\varepsilon)$

let $N_0 = \max\{n_1(\varepsilon), n(\varepsilon)\}$

$|a-b| = |a - a_n + a_n - b| \leq |a - a_n| + |a_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \; \forall n \geq N_0$

$|a - b| < \varepsilon \quad \forall n \geq N_0$

$\Rightarrow |a - b| = 0$

$a = b$

Sequence diverging to $+\infty$ and $-\infty$

Def A sequence $\{a_n\}$ is said to diverge to $+\infty$ i.e $\lim_{n \to \infty} a_n = +\infty$ if for any given $\Delta \in \mathbb{R}$ $\exists n_0 \in \mathbb{N}$ such that $n \geq n_0 \Rightarrow a_n > \Delta$

A sequence $\{a_n\}$ is said to diverge to $-\infty$ i.e $\lim_{n \to \infty} a_n = -\infty$ if for any ~~diverge to~~ given $\Delta \in \mathbb{R}$ $\exists n_0 \in \mathbb{N}$ such that $n \geq n_0 \Rightarrow a_n < \Delta$.

Theorem: suppose that $\{a_n\}$ and $\{b_n\}$ are sequence such that $a_n \leq b_n \; \forall n \in \mathbb{N}$

① If $\lim_{n \to \infty} a_n = +\infty$ then $\lim_{n \to \infty} b_n = +\infty$

② If $\lim_{n \to \infty} b_n = -\infty$ then $\lim_{n \to \infty} a_n = -\infty$

proof

DATE

(i) $a_n \le b_n \quad \forall n \in \mathbb{N}$

$\lim_{n\to\infty} a_n = +\infty \Rightarrow \forall \Delta \in \mathbb{R}$ given $\exists n_0 \in \mathbb{N}$ s.t

$n > n_0 \Rightarrow a_n > \Delta$

but $b_n \ge a_n > \Delta \Rightarrow b_n > \Delta$

$\therefore b_n \to +\infty$

(2)

Exercise

show that (i) $\{2^n\}$ diverges (ii) $\{\sqrt{n-7}\}$ diverges

Subsequence

Def: let $f: \mathbb{N} \to X$ be a sequence of a set X denoted by $\{a_n\}_{n=1}^{\infty}$ let $n: \mathbb{N} \to \mathbb{N}$ be strictly increase one to one function which is defined by

$n(k) = n_k \in \mathbb{N} \quad \forall k \in \mathbb{N}$

by a subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ of $\{a_n\}_{n=1}^{\infty}$ we mean the composition map

$f \circ n : \mathbb{N} \xrightarrow{n} \mathbb{N} \xrightarrow{f} X$

i.e $(f \circ n)(k) = a_{n_k} \quad k \in \mathbb{N}$

Informally one can define a subsequence as follows.

DATE

let $\{a_n\}$ be a sequence in X and let $\{n_k\}$ be any sequences of $\mathbb{N}$.

$$\frac{n^2+2}{n^3}$$

$a_n = (1+1/n)^n$, $b_n = (1+1/n)^{n+1}$

$b_n \geq a_n$

$b_n - a_n \geq 0$

claim $(1+1/n)^n \leq 3$ $\qquad \frac{(1+1/n)^n}{n} \leq \frac{3}{n}$

$$(1+1/n)^n = 1 + \frac{n}{n} + \frac{1}{n^2}\frac{n(n-1)}{2!} + \frac{n(n-1)(n-2)}{3!\,n^3} + \frac{n(n-1)(n-2)(n-3)}{4!\,n^4} + \cdots$$

$$\cancel{1+\frac{1}{2!}(n-1)}\quad 1+1+\frac{1}{2!}\frac{(n-1)}{n} + \frac{1}{3!}\frac{(n-1)}{n}\frac{(n-2)}{n} + \frac{1}{4!}\frac{(n-1)}{n}\frac{(n-2)}{n}\frac{(n-3)}{n}$$

$$\leq 2 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots \leq 2 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$$

$(2+1 = 3)$

$2^{n-1} \leq n!$ $\forall n \in \mathbb{N}$

$\frac{1}{2^{n-1}} \geq \frac{1}{n!}$ $\qquad$ G.P $a = 1/2$ $r = 1/2$

$r < 1$ $\quad S_\infty = \frac{a}{1-r} = \frac{1/2}{1-1/2} = 1$

Limit superior and limit Inferior

let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence of $\mathbb{R}$ consider number M_n defined as follows.

$M_n = \sup_n \{a_n\ a_{n+1}\ a_{n+2} \cdots\} = \sup_{i \geq n} a_i$

$M_{n+1} = \sup_n \{a_{n+1}, a_{n+2}, a_{n+3} \cdots\} = \sup_{i \geq n+1} a_i$

$A = \{a_n\ a_{n+1}\ a_{n+2} \cdots\}$

$B = \{a_{n+1}\ a_{n+2}\ a_{n+3} \cdots\}$

$B \subset A$

$\sup A \geq \sup B \Rightarrow M_{n+1} \leq M_n$ $\quad \forall n \in \mathbb{N}$

((?) (( ) (?)

DATE

$\Rightarrow M_n$ is M-D sequence and bounded

why? because a_n is bounded. Hence M_n converges

$\therefore \lim_{n\to\infty} M_n = \inf_n M_n$

lim sup

$\lim_{n\to\infty} M_n = \inf_n M_n = \inf_{n\geq 1} \left(\sup_{i\geq 1} a_i\right) = \beta$

β is the limit superior of a sequence $\{a_n\}_{n\geq 1}$

$\beta = \lim_{n\to\infty} \sup a_n = \lim_{n\to\infty} a_n = \inf_{n\geq 1} \left(\sup_{i\geq n} a_i\right)$

Limit Inferior

let $\{a_n\}_{n=1}^{\infty}$ be bounded seq. of $\mathbb{R}$ consider the number m_n defined as follows

$m_n = \inf_n \{a_n \; a_{n+1} \; a_{n+2} - - - \}$ $\inf_{i\geq n} a_i$

$m_{n+1} = \inf_n \{a_{n+1} \; a_{n+2} \; a_{n+3} - - - - \}$ $\inf_{i\geq n+1} a_i$

$A = \{a_n \; a_{n+1} \; a_{n+2} - - - - \}$

$B = \{a_{n+1} \; a_{n+2} \; a_{n+3} - - - - \}$

$B \subset A$

$\inf_n A \leq \inf B$

$m_n \leq m_{n+1} \; \forall n \geq 1 \Rightarrow m_n$ is M.I and is bounded

Hence m_n converges

$\therefore \lim_{n\to\infty} m_n = \sup_n m_n = \sup_n \left(\inf_{i\geq n} a_i\right) = \alpha$

α is the limit Inferior of $\{a_n\}$ and is ~~bounded~~ denoted by $\alpha = \lim_{n\to\infty} \inf a_n = \lim_{n\to\infty} a_n = \sup_{n\geq 1} \left(\inf_{j\geq n} a_i\right)$

Note

$\lim_{n\to\infty} \sup a_n = \beta \Rightarrow$ given $\varepsilon > 0 \; \exists n(\varepsilon) \in \mathbb{N}$

s.t $a_n \leq \beta + \varepsilon$. ~~$\forall n(\varepsilon)$~~ $\forall n \geq n(\varepsilon)$

$\beta \lim_{n\to\infty} M_n$

Tuesday

3^{rd} April 2—4 pm.

DATE

$\beta = \lim_{n\to\infty} M_n$ means $\forall\ \varepsilon > 0$ given $\exists\ n(\varepsilon) \in \mathbb{N}$

such that $|M_n - \beta| < \varepsilon \quad \forall n \geq n(\varepsilon)$

$\beta - \varepsilon < M_n < \beta + \varepsilon \quad \forall n \geq n(\varepsilon)$

$\sup_{i \geq n} a_i < \beta + \varepsilon \quad \forall n \geq n(\varepsilon)$

$a_n \leq \sup_{i\geq n} a_i < \beta + \varepsilon \quad \forall n \geq n(\varepsilon)$

$\Rightarrow a_n < \beta + \varepsilon \quad \forall n \geq n(\varepsilon)$

Theorem: A bounded sequence $\{a_n\}_{n=1}^{\infty}$ is convergence to $a \in \mathbb{R}$ if and only if

$\liminf_{n\to\infty} a_n = a = \limsup_{n\to\infty} a_n$

proof exercise

for each of the following sequences find the limit superior and the limit inferior

(i) $a_n = \{(-1)^n + \frac{1}{n}\}_{n=1}^{\infty}$ (ii) $a_n = n \sin\left(\frac{n\pi}{2}\right)$

(iii) $(n(n-1))_{n=1}^{\infty}$ (iv) $\{(-1)^n\}_{n=1}^{\infty}$

Check the convergences or divergences of the fff (a) $n + \sin(n)$ (b) $\frac{\sin(n^4)}{\sqrt{n}}$ (c)

(c) $-n + (-1)^n$ (d) $n \sin(\frac{1}{n})$

show that $\cos\frac{n\pi}{3}$ donot converges

$|a_n - l| < \varepsilon$

$(a_n)_{n=b}$ $a_n \to r$

$b_n \subset a_n$

$\sup = b_n = $ 1. ~~infinite~~ ~~term~~ 2 ~~upperb~~

DATE 10/4/2018

series of non-negative real numbers

Def: A series $\sum_{n=1}^{\infty} a_n$ of real numbers is defined as the double sequence $\{a_n, S_n\}$ satisfying the following conditions $S_n = \sum_{i=1}^{n} a_i$ where, $a_n = S_n - S_{n-1}$

$S_n = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$

$S_{n-1} = a_1 + a_2 + a_3 + \cdots + a_{n-1}$

a_n is the general term of the series and

S_n is the sequence of n^{th} partial sum

Def: A series $\sum_{i=1}^{\infty} a_i$ is called convergent if and only if its sequence of n^{th} partial sum $\{S_n\}$ converges to a then we note $\sum_{i=1}^{\infty} a_i = a$

eg consider the geometric series $\sum_{i=1}^{\infty} \left(\frac{2}{3}\right)^i$ converges to 2

$$S_n = \sum_{i=1}^{n} \left(\frac{2}{3}\right)^i = \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \cdots + \left(\frac{2}{3}\right)^n$$

consider g.p $S_n = a\left(\frac{1-r^n}{1-r}\right)$

$$S_n = \frac{2 \cdot 3}{3}\left(1 - \left(\frac{2}{3}\right)^n\right)$$

$$\lim_{n \to \infty} S_n = 2$$

$$\therefore \sum_{i=1}^{\infty} \left(\frac{2}{3}\right)^i = 2$$

$\sum_{i=1}^{n-1}$ S_{n-1} ∞

Determine wether the series converges

$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots +$

$$\sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i$$

$\frac{a_{n+1}}{a_n} = r =$ common ratio

~~2 3 4 5~~ $2, 2^2, 2^3, 2^4 -$

$S_n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \left(\frac{1}{2}\right)^n$

$S_n = 2\left(1 - \left(\frac{1}{2}\right)^n\right) \xrightarrow[n\to\infty]{} 2$

$\sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i = 2$

Determine whether the series $\sum_{i=1}^{\infty} \frac{1}{(3i-1)(3i+2)}$ converges.

$= \sum_{i=1}^{n} \frac{1}{(3i-1)(3i+2)}$

$\frac{1}{(3n-1)(3n+2)} = \frac{A}{3n-1} + \frac{B}{3n+2}$

$1 = A(3n+2) + B(3n-1)$

$n = -\frac{2}{3}$ $\quad 1 = A\left(3\left(\frac{-2}{3}\right)+2\right) + B\left(3\left(\frac{-2}{3}\right)-1\right)$

$1 = -3B \implies B = -\frac{1}{3}$

$n = \frac{1}{3}$ $\quad 1 = A\left(3\left(\frac{1}{3}\right)+2\right) + B\left(3\left(\frac{1}{3}\right)-1\right)$

$1 = 3A$

$\implies A = \frac{1}{3}$

$\frac{1}{(3n-1)(3n+2)} = \frac{A}{3n-1} + \frac{B}{3n+2} = \frac{1}{3}\left(\frac{1}{3n-1} - \frac{1}{3n+2}\right)$

$\therefore S_n = \frac{1}{3}\sum_{i=1}^{n}\left(\frac{1}{3i-1} - \frac{1}{3i+2}\right)$

$S_n = \frac{1}{3}\left(\frac{1}{2} - \cancel{\frac{1}{5}} + \cancel{\frac{1}{5}} - \cancel{\frac{1}{8}} + \cancel{\frac{1}{8}} - \cancel{\frac{1}{11}} + \cancel{\frac{1}{11}} - \cancel{\frac{1}{14}} + \cancel{\frac{1}{14}} + \cdots + \cancel{\frac{1}{3n-1}} - \frac{1}{3n+2}\right)$

$S_n = \frac{1}{3}\left(\frac{1}{2} - \frac{1}{3n+2}\right) \xrightarrow[n\to\infty]{} \frac{1}{6}$

$\therefore \sum_{i=1}^{\infty} \frac{1}{(3i-1)(3i+2)} = \frac{1}{6}$

$$\frac{1}{(i+1)!} = \frac{1}{i!} - \frac{1}{(i+1)!}$$

DATE

prove that $\sum_{i=1}^{\infty} \frac{i}{(i+1)!}$ Converges to 1

$$S_n = \sum_{i=1}^{n} \frac{i}{(i+1)!} = \sum_{i=1}^{\infty} \left(\frac{1}{i!} - \frac{1}{(i+1)!}\right)$$

$$S_n = 1 - \frac{1}{2!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{4!} + \cdots + \frac{1}{(n-1)!} - \frac{1}{n!}$$

$$S_n = 1 - \frac{1}{n!} \xrightarrow{n \to \infty} 1$$

$$\therefore \sum_{i=1}^{\infty} \frac{i}{(i+1)!} = 1$$

Exercise

show that the series $\sum_{i=1}^{\infty} \frac{i-1}{(i+1)(i+2)(i+3)}$ Converges to $\frac{1}{6}$

Proposition: A neccessary Condition for a Series $\sum_{i=1}^{\infty} a_i$ to be convergent is that $\lim_{n \to \infty} a_n = 0$

The integral test Theorem

The integral test: If f is a non-negative decrease integrable function such that $f(x) = a_n$ $\forall n \in [a, \infty]$ then the series $\sum_{n=a}^{\infty} a_n$ and the integral $\int_a^{\infty} f(t)\, dt$ Converges or diverges together.

proof

$f(n) \geq f(t) \geq f(n+1)$ $\qquad n \leq t \leq n+1$

$\int f(n) \geq \int f(t) \geq \int f(n+1)$

$a_n \geq \int f(t) \geq a_{n+1}$

Corollary to that is harmonic series $\sum \frac{1}{n}$

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P test

Example: show that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p>1$ and diverges if $p \leq 1$.

Solution

If $p<0$ then $\frac{1}{n^p} \not\to 0$, $\sum \frac{1}{n^p}$ diverges.

Let $f(t) = \frac{1}{t^p}$

$I(N) = \int_1^N \frac{1}{t^p} dt = \ln t \Big|_1^N$ if $p=1$

$\frac{t^{1-p}}{1-p}\Big|_1^N$ if $p \neq 1$

$$\begin{cases} \ln N & p=1 \\ \frac{N^{1-p}-1}{1-p} & p\neq 1 \end{cases}$$

$$\lim_{N\to\infty} I(N) = \begin{cases} \infty & \text{if } 0 \leq p \leq 1 \\ \frac{-1}{1-p} & p>1 \end{cases}$$

Exercise

Using the integral test, test the convergent or otherwise of the following series:

(1) (i) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ (ii) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ (iii) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$, $p \in \mathbb{R}$

(2) Show that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges to a sum less than 2

(3) Show that the series $\sum_{n=2}^{\infty} \left(\frac{1}{n(\ln n)^p}\right)$ $p>0$ converges if $p>1$ and diverges if $p \leq 1$

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The Comparison test. 17/4/2018

- suppose that $0 \le a_n \le k b_n$ for some $k > 0$ and $\forall n \ge n_0$ for some $n_0 \in \mathbb{N}$. and that $\sum_{n=1}^{\infty} b_n$ Converges then $\sum_{n=1}^{\infty} a_n$ Converges.

- suppose that $a_n \ge k b_n$ for $n \ge n_0$ for some $k > 0$ and $n_0 \in \mathbb{N}$. And that $\sum_{n=1}^{\infty} b_n$ diverges then $\sum_{n=1}^{\infty} a_n$ diverges

— Proof as Exercise

Limit Comparison test

Theorem: suppose that $a_n > 0$ $b_n > 0$ $\forall n \ge n_0 \in \mathbb{N}$

And $\lim_{n \to \infty} \frac{a_n}{b_n} = L$

then if $L = 0$ $\sum_{n=1}^{\infty} b_n$ Converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ Converges

if $L \in (0, \infty)$ $\sum_{n=1}^{\infty} a_n$ $\sum_{n=1}^{\infty} b_n$ converges/diverges together

if $L = \infty$ $\sum_{n=1}^{\infty} b_n$ diverges $\Rightarrow \sum_{n=1}^{\infty} a_n$ diverges

Proof

If $L = 0$ $\forall \varepsilon > 0$ $\exists n_0 \in \mathbb{N}$ $\forall n \ge n_0$ $\left|\frac{a_n}{b_n} - 0\right| < \varepsilon$

$\Rightarrow a_n < \varepsilon b_n$ $\forall n \ge n_0$

by Comparison test

$\sum_{n=1}^{\infty} b_n$ Converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ Converges.

if $L \in (0, \infty)$ let $\varepsilon = \frac{L}{2}$

$\forall n_0 \in \mathbb{N}$. $\forall n \ge n_0$ $\left|\frac{a_n}{b_n} - L\right| < \frac{L}{2}$

$f(x) = \int$

$-\frac{L}{2}+L < \frac{a_n}{b_n} < \frac{L}{2}+L$

$\frac{L}{2} b_n < a_n < \frac{3L}{2} b_n$

Example: Verify the convergence or divergence of the ffg

(a) $\sum_{n=1}^{\infty} \frac{n}{n^4-3}$

Solution

$a_n = \frac{n}{n^4-3} < \frac{n}{n^4-\frac{n^4}{2}}$ for $\frac{n^4}{2} > 3$ ($n \geq 2$)

$< n^3$

$\underbrace{\frac{n}{n^4-3}}_{a_n} < \underbrace{\frac{2}{n^3}}_{b_n}$ $\forall n \geq 2$

$0 < a_n \leq k b_n \quad \forall n \geq n_0$

$\sum b_n = \sum \frac{1}{n^3}$ converges

Therefore by comparison test $\sum_{n=1}^{\infty} a_n$ converges

$b_n = \frac{n}{n^4} = \frac{1}{n^3}$ $\quad \frac{a_n}{b_n} = \frac{n}{n^4-3} \cdot n^3 = \frac{1}{1-\frac{3}{n^4}} \longrightarrow 1$

$\lim_{n\to\infty} \frac{a_n}{b_n} = 1$ by limit comparison test since $\sum b_n$ converg

by the $\sum a_n$ converges

(2) $\sum_{n=1}^{\infty} \frac{n^2}{e^{n^2}}$, $\quad a_n = \frac{n^2}{e^{n^2}} = \frac{n^2}{1+n^2+\frac{n^4}{2!}+\frac{n^6}{3!}+\cdots}$

$< \frac{n^2}{1+n^2+\frac{n^4}{2!}} < \frac{n^2}{n^2+\frac{n^4}{2!}}$

$= \frac{1}{1+\frac{n^2}{2}} < \frac{2}{n^2}$

$\sum \frac{1}{n^2}$ converges $\Rightarrow \sum a_n$ converges

$\frac{\sqrt[n]{n}}{2}$

DATE

(3) $\sum_{n=1}^{\infty} \frac{n^2}{n^3-3}$ let $b_n = \frac{n^2}{n^3} = \frac{1}{n}$

$\frac{a_n}{b_n} = \frac{n^3}{n^3-3} = \frac{1}{1-\frac{3}{n^3}} \xrightarrow{n \to \infty} 1$

$\therefore \sum_{n=1}^{\infty} a_n$ diverges

Exercises use Comparison test to verify the convergence or otherwise of the following

(a) $\sum_{n=1}^{\infty} \frac{3^n+1}{4^n-1}$ (b) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{2n^2+1}$ (c) $\sum_{n=1}^{\infty} \frac{n+1}{3n^2+1}$

(d) $\sum_{n=1}^{\infty} \frac{2^n-1}{5^n-1}$ (e) $\sum_{n=1}^{\infty} \frac{10^n}{n!}$ (f) $\sum_{n=1}^{\infty} n^{-\frac{1}{2n}}$

Cauchy Root test

let $\sum_{n=1}^{\infty} a_n$ be a series of non-negative terms and suppose $\lim_{n \to \infty} \sqrt[n]{a_n} = l$

If $l < 1$ then $\sum a_n$ converges

If $l > 1$ then $\sum a_n$ diverges

If $l = 1$ nothing to conclude

proof Exercise

Examples discuss the convergence of the following series

(1) $\sum_{n=1}^{\infty} \frac{n}{2^n}$, $a_n = \frac{n}{2^n}$ $\sqrt[n]{a_n} = \left(\frac{n}{2^n}\right)^{\frac{1}{n}}$

$= \frac{\sqrt[n]{n}}{2}$

$\lim_{n \to \infty} \sqrt[n]{a_n} = \lim_{n \to \infty} \frac{\sqrt[n]{n}}{2} = \frac{1}{2} < 1$

$\therefore \sum_{n=1}^{\infty} a_n$ Converges

Exercise Test the convergence or otherwise of the following

(i) $\sum_{n=1}^{\infty} \frac{1}{3^n (n)}$ (ii) $\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{e}{\pi-1}\right)^n$ (iii) $\sum_{n=1}^{\infty} n^2 e^{an}$

(iv) $\sum_{n=1}^{\infty} \frac{1}{2^{2n}+(-1)^n}$ (v) $\sum_{n=1}^{\infty} \frac{1}{n^{3+(-1)^n}}$ (vi) $\sum_{n=1}^{\infty} \frac{\log n}{2^n}$

D'Alembert's ratio Test

Theorem Let $\sum_{n=1}^{\infty} a_n$ be a series of non negative terms and suppose $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l$

then ~~if l<1~~

if $l<1$ $\sum a_n$ converges

if $l>1$ $\sum a_n$ diverges

if $l=1$ nothing to conclude

~~proof~~ as exercise

Example: test the convergence of the following

(i) $\sum_{n=1}^{\infty} \frac{10^n}{n}$, $a_n = \frac{10^n}{n}$ $a_{n+1} = \frac{10 \cdot 10^n}{n+1}$

$\frac{a_{n+1}}{a_n} = \frac{10 \cdot \cancel{10^n}}{n+1} \cdot \frac{n}{\cancel{10^n}} = \frac{10n}{n+1} \xrightarrow{n\to\infty} 10$

$10 > 1$

$\therefore \sum a_n$ diverges

$\sum_{n=1}^{\infty} \frac{a^n}{n^2}$ $a>0$ $b_n = \frac{a^n}{n^2}$, $b_{n+1} = \frac{a \cdot a^n}{(n+1)^2}$

$\frac{b_n}{b_{n+1}} = \frac{a^n}{n^2} \cdot \frac{(n+1)^2}{a^n \cdot a} = \left(\frac{n+1}{n}\right)^2 \frac{1}{a}$

$\frac{b_n}{b_{n+1}} \xrightarrow{n\to\infty} \frac{1}{a}$

$2^3 = (2)^{3/1}$

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for $0 < a < 1$ $\quad \frac{1}{a} > 1$

$\sum a_n$ converges

for $a > 1$ $\quad \frac{1}{a} < 1$

$\sum a_n$ diverges

Exercise: Test the convergence or otherwise of the following:

(i) $\sum_{n=1}^{\infty} \frac{3n+1}{n-2}$ (ii) $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$ (iii) $\sum_{n=1}^{\infty} \frac{n!}{n!+3}$

(iv) $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^3}$

Alternating Series

Def: A series in which the terms and alternately positive and negative is called an alternating series. ~~dt~~ It is sume term are $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$

where $a_n > 0 \ \forall n$

$1 - 2 + 3 - 4 + 5 - 6 + \cdots + (-1)^{n-1} n + \cdots$